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The criterion for quantum teleportation of fock states

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Abstract

The quantum teleportation of fock states is formulated through the state evolution by means of the general theory of quantum-mechanical measurement in the coherent state basis. The criterion for evaluating the quantum teleportation of fock states is established, which is different from that of teleportation of coherent states.

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1. Introduction

Quantum teleportation is a method via which quantum information encoded in a quantum state can be passed through a classical channel and successfully retrieved at a distant location. It represents the basic building block of future quantum communication networks between parties and has attracted considerable interest in quantum information theory.

Quantum teleportation was originally proposed for the teleportation of single particles [1], the experimental realizations with photons have been made for demonstration the teleportation of discrete single photon polarization states [2,3]. Later, both the theory and the experiment show that it can be extended for teleportation of coherent states of continuous electromagnetic fields [4–6]. The next challenge is the teleportation of truly non-classical states [7], the first introduction of teleportation of non-classical states is the entanglement swapping in the discrete-variable systems [8–12], it has also been investigated for the unconditional teleportation of continuous variable entanglement [13–15], where both the Einstein–Podolsky–Rosen (EPR) pairs and the teleported input non-classical state are provided by the two-mode squeezed vacuum states.

It is well known that the capacity of a photon channel can be improved in the transmission of quantum information by the use of non-classical states [16]. Therefore, the transfer of non-classical states is of interest [17]. Fock state as a perfect amplitude-squeezed state provides the maximum channel capacity in optical communication [18]. Thus the teleportation of fock states may be preferable for improving the information capacity

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in quantum information science. In [16], the teleportation of a one photon fock state was theoretically analyzed, and the properties of the output state was thoroughly investigated.

In the real teleportation scheme for continuous variable teleportation, the entanglements of the EPR resource are always imperfect since it would require infinite energy. The criteria for evaluating the efficiency of teleportation of quantum states of light fields are therefore presented in various methods [5,19–21]. Some types of teleportation devices have been investigated for improvement of the degree of EPR entanglement and the fidelity of the teleported state [22–25].

In this paper, the criterion for characterizing a threshold for the appearance of non-classical effects in any number fock state teleportation process is formulated by fidelity. For the experiment simplicity, the teleportation process we consider here is similar to that described by Braunstein and Kimble [5] The EPR resource we used is the two-mode squeezed vacuum state, the Bell detection is performed by the joint measurement of quadrature amplitudes on input mode and one half of EPR beams after a 50:50 beamsplitter. In Section 2 the teleportation of any quantum state is analyzed using the general theory of quantum-mechanic measurement. The criterion for teleportation of fock states is established in Section 3. The most interesting result is that the fidelity for evaluating the efficiency of the teleportation of fock state is very different from that of coherent state teleportation.

2. The state evolution for teleportation of quantum states

The first step of teleportation is preparation of EPR quantum entangled pairs. In the continuous variable teleportation scheme, the EPR entangled state is a two-mode squeezed vacuum state, the density matrix of the squeezed state can be expanded with the fock state basis under the limit of infinite photon-numbers,

$$\rho_{1,2} = (1-\lambda^2)\sum_{n}^{\infty}\sum_{n'}^{\infty}(-\lambda)^{n+n'}|n,n\rangle_{1,2}\langle n',n'|, \qquad (1)$$

where the indices 1, 2 denote the two modes of the squeezed state, $\lambda = \tanh r$ $(0 < \lambda < 1)$ is a parameter quantifying the strength of squeezing. In the limit $r \to \infty$, i.e., $\lambda \to 1$, the squeezed vacuum state is maximally squeezed, in this case the two-mode squeezed state is the analog of ideal EPR pairs with entanglements between quadrature phase amplitudes. For finite squeezing $(\lambda < 1)$, it shows the non-classical correlations properties of non-ideal EPR pairs. $\lambda = 0$ means no entanglement for the two-mode squeezed state.

For the input quantum states, the density operator of the unknown input quantum state $|\Psi\rangle_{\rm in}$ takes the form:

$$\hat{\rho}_{\rm in} = |\Psi\rangle_{\rm in}\langle\Psi|. \qquad (2)$$

The statistical density operator of the initial state combining the input unknown state with the EPR pairs is:

$$\hat{\rho}_0 = \hat{\rho}_{\rm in}\otimes\hat{\rho}_{1,2}. \qquad (3)$$

Now we consider the second step of teleportation. First the input state is combined with a half of EPR pairs at the 50:50 beamsplitter, then the outputs are measured by the homodyne detections.

As well known, the lossless 50:50 beamsplitter combining the input and a half of EPR pairs is described by the unitary operator $\hat{U}$ [26],

$$\hat{U} = {\rm e}^{(\pi/4)\left(\hat{a}_{\rm in}\hat{a}_1^{+}-\hat{a}_1\hat{a}_{\rm in}^{+}\right)}, \qquad (4)$$

for simplicity, we have ignored the phase-shifts introduced by the beamsplitter in Eq. (4) which shows that a lossless beamsplitter is characterized by the generators of the SU(2) Lie group.

In the Schrödinger picture, the density matrix of the output state of the beamsplitter is obtained through the unitary transformation of the density matrix $\hat{\rho}_0$ of the initial state [27,28]

$$\hat{\rho}_{\rm BS} = \hat{U}^{+}\hat{\rho}_0\hat{U}. \qquad (5)$$

Using the decomposition formula of the SU(2) Lie algebra [29],

$$\hat{U}^{+} = {\rm e}^{\hat{a}_{\rm in}^{+}\hat{a}_1}(\sqrt{2})^{\hat{a}_{\rm in}^{+}\hat{a}_{\rm in}-\hat{a}_1^{+}\hat{a}_1}\,{\rm e}^{-\hat{a}_{\rm in}\hat{a}_1^{+}}, \qquad (6)$$

and substituting Eqs. (1)–(3) and (6) into Eq. (5), the density matrix of the output state of the beamsplitter takes the form in the coherent state basis:

$$\hat{\rho}_{\rm BS} = \frac{\sqrt{1-\lambda^2}}{\pi^3} \int\int\int {\rm d}^2\alpha\, {\rm d}^2\beta\, {\rm d}^2\gamma {\rm e}^{-\frac{1}{2}|\alpha|^2} {\rm e}^{-\frac{3}{4}|\beta|^2} \times {\rm e}^{-\frac{1}{2}|\gamma|^2} {\rm e}^{-\gamma\beta^*} {\rm e}^{\alpha^*\beta/\sqrt{2}} {\rm e}^{\sqrt{2}\alpha^*\gamma} {\rm e}^{-\lambda\beta^* a_2} \langle\gamma|\Psi\rangle_{\rm in}|\alpha\rangle_{\rm in} \times |\beta/\sqrt{2}\rangle_1 |0\rangle_2 \otimes {\rm h.c}, \tag{7}$$

where we have used the following relations:

$$(\sqrt{2})^{-\hat{a}_1^+\hat{a}_1}|\beta\rangle_1 = {\rm e}^{-\frac{1}{4}|\beta|^2}|\beta/\sqrt{2}\rangle_1, \quad (\sqrt{2})^{\hat{a}_{\rm in}^+\hat{a}_{\rm in}}|\gamma\rangle_{\rm in} = {\rm e}^{\frac{1}{2}|\gamma|^2}|\sqrt{2}\gamma\rangle_{\rm in}, \tag{8}$$

$$|n\rangle_2 = \frac{(\hat{a}_2^+)^n}{\sqrt{n!}}|0\rangle_2, \tag{9}$$

$$\sum_n^\infty \frac{(-\lambda\beta^*\hat{a}_2^+)^n}{n!} = {\rm e}^{-\lambda\beta^*\hat{a}_2^+}, \tag{10}$$

$$\langle\beta|n\rangle_1 = {\rm e}^{-\frac{1}{2}|\beta|^2}\frac{(\beta^*)^n}{\sqrt{n!}}. \tag{11}$$

For the Bell-operators measurement, the in-phase quadrature $\hat{X}$ of one output beam of the beamsplitter and the out-of-phase quadrature $\widehat{Y}$ of the other one are simultaneously measured with two homodyne detection systems. In the Schrödinger picture the measured outcomes are corresponding to the operators:

$$\hat{X} = \left(\hat{a}_{\rm in} + \hat{a}_{\rm in}^+\right)/2, \quad \widehat{Y} = -{\rm i}\left(\hat{a}_1 - \hat{a}_1^+\right)/2. \tag{12}$$

According to the general theory of quantum-mechanical measurement, the quantum measurement is mathematically described by a positive operator-valued measure (POVM) including a projection operator [30]. The positive operator-valued continuous measurement of two quadrature phases of X and Y are given by:

$$\Pi_{\rm in}(X) = |X\rangle_{\rm in}\langle X|, \quad \Pi_1(Y) = |Y\rangle_1\langle Y|. \tag{13}$$

The states $|X\rangle_{\rm in}$ and $|Y\rangle_1$ are the eigenstates of the quadrature components satisfying the completeness relations: $\int_{-\infty}^{\infty}|X\rangle_{\rm in}\langle X|\,{\rm d}X = \hat{I}$, $\int_{-\infty}^{\infty}|Y\rangle_1\langle Y|\,{\rm d}Y = \hat{I}$. When the values are measured, the other half of EPR pairs collapses into a conditional state. The normalized density matrix of the conditional output state is expressed by:

$$\hat{\rho}_2(X,Y) = \frac{{\rm Tr}_{\rm in,1}\left\{\hat{\rho}_{\rm BS}\Pi_{\rm in}(X)\Pi_1(Y)\right\}}{P(X,Y)}, \tag{14}$$

where ${\rm Tr}_{\rm in,1}$ stand for the trace operations with respect to the input state and half of EPR pairs. $P(X,Y)$ is the probability distribution of the measured results:

$$P(X,Y) = {\rm Tr}_2\,{\rm Tr}_{\rm in,1}\left\{\hat{\rho}_{\rm BS}\Pi_{\rm in}(X)\Pi_1(Y)\right\}, \tag{15}$$

where ${\rm Tr}_2$ stands for the trace operations with respect to the other half of EPR pairs.

To calculate the conditional output state of the other half of EPR pairs, we use the expressions of the coherent state in the momentum and position basis to give the wave function of the quadrature components of the coherent state:

$$\langle X|\alpha\rangle_{\rm in} = \left(\frac{2}{\pi}\right)^{1/4} \times \exp\left\{-X^2 + 2\alpha X - \frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^2\right\}, \quad \langle Y|\beta/\sqrt{2}\rangle_1 = \left(\frac{2}{\pi}\right)^{1/4} \times \exp\left\{-Y^2 - {\rm i}\sqrt{2}\beta Y - \frac{1}{4}|\beta|^2 + \frac{1}{4}\beta^2\right\}. \tag{16}$$

Substituting Eqs. (7), (13) and (16) into Eq. (14) and integrating out the parameters α and β, Eq. (14) becomes:

$$\hat{\rho}_2(X,Y) = \frac{\sqrt{2/\pi}\sqrt{1-\lambda^2}\,{\rm e}^{-(X^2+Y^2)}|\Phi\rangle \otimes {\rm h.c}}{P(X,Y)}, \tag{17}$$

and hence

$$|\Phi\rangle = f(\gamma,\Psi)\hat{D}(\lambda[\gamma - \sqrt{2}(X - {\rm i}Y)])|0\rangle_2,$$
$$f(\gamma,\Psi) = \frac{1}{\pi}\int {\rm d}^2\gamma {\rm e}^{-\frac{1}{2}|\gamma|^2} {\rm e}^{\sqrt{2}\gamma(X+{\rm i}Y)} {\rm e}^{\frac{1}{2}\lambda^2\left|\gamma-\sqrt{2}(X-{\rm i}Y)\right|^2}\langle\gamma|\Psi\rangle_{\rm in},$$

where the displacement operator is

$$\hat{D}(\lambda[\gamma - \sqrt{2}(X - {\rm i}Y)]) = {\rm e}^{-\frac{1}{2}\lambda^2|\gamma-\sqrt{2}(X-{\rm i}Y)|^2} {\rm e}^{\lambda[\gamma-\sqrt{2}(X-{\rm i}Y)]\hat{a}_2^+} {\rm e}^{\lambda[\gamma-\sqrt{2}(X-{\rm i}Y)]\hat{a}_2}.$$

The probability distribution measured for values X and Y is given by:

$$P(X,Y) = (2/\pi)(1-\lambda^2)\,\mathrm{e}^{-2(X^2+Y^2)} f(\gamma,\Psi) f^*(\gamma',\Psi). \tag{18}$$

We can easily prove that for all observables, the probability is normalized:

$$(2/\pi)(1-\lambda^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}X\,\mathrm{d}Y\,\mathrm{e}^{-2(X^2+Y^2)} f(\gamma,\Psi) \times f^*(\gamma',\Psi) = 1. \tag{19}$$

According to the protocol of teleportation, the last step is to perform a unitary transformation on the other half of the EPR pairs with the above-measured results to mimic the input unknown state. This transformation is carried out by a displacement operator $\hat{D}(\sqrt{2}g(X-\mathrm{i}Y)])$. Therefore, the state described by Eq. (17) is transformed into:

$$\hat{\rho}_2^{\mathrm{out}} = \frac{\sqrt{2/\pi}\sqrt{1-\lambda^2}\,\mathrm{e}^{-(X^2+Y^2)}\hat{D}(\sqrt{2}g(X-\mathrm{i}Y))|\Phi\rangle \otimes \mathrm{h.c}}{P(X,Y)}, \tag{20}$$

where g represents a normalized classical gain for the transformation from classical measured infinitesimal values X and Y to complex field amplitudes $X - \mathrm{i}Y$.

Consider the ideal condition of infinite squeezing and the unity classical transformation gain $g = 1$, $\lambda = 1$, we can obtain the density matrix of the conditional output state of the other half of EPR pairs:

$$\hat{\rho}_2^{\mathrm{out}} = \frac{\sqrt{2/\pi}\sqrt{1-\lambda^2}\{\frac{1}{\pi}\int \mathrm{d}^2\gamma|\gamma\rangle\langle\gamma|\Psi\rangle\} \otimes \mathrm{h.c}}{P_{\mathrm{ideal}}(X,Y)} = |\Psi\rangle\langle\Psi| = \hat{\rho}_{\mathrm{in}}. \tag{21}$$

It shows that the input unknown quantum state is perfectly teleported to the receiving station of the other half of EPR pairs being under the limit of infinite squeezing and unity classical transformation gain for the lossless measurement system.

3. The criterion for teleportation of fock states

On the other hand, for the case of $\lambda < 1$, which corresponds to the non-ideal EPR pairs (i.e., imperfect squeezed state), the output state is a conditional output state of particular measurement outcomes. When we ignore the outcomes or perform the measurements X and Y from $-\infty$ to ∞, the resulting output state behaves like a mixture of the unnormalized density matrix elements,

$$\hat{\rho}_T = \sqrt{(2/\pi)(1-\lambda^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}X\,\mathrm{d}Y\,\mathrm{e}^{-(X^2+Y^2)} \times \hat{D}(\sqrt{2}g(X-\mathrm{i}Y))|\Phi\rangle \otimes \mathrm{h.c.} \tag{22}$$

Indeed when $\lambda < 1$ and $g \neq 1$, Eq. (22) shows that the output state overlaps with the input state partially. The extent of similarity between the output state and the input fock state $|m\rangle$ is quantitatively evaluated with the averaged fidelity F [31]:

$$F = \langle m|\hat{\rho}_T|m\rangle. \tag{23}$$

We know that $\lambda = 0$ is the classical limit of EPR entanglement, thus the threshold fidelity of quantum teleportation without entanglement is established by setting $\lambda = 0$ and $g = 1$ in Eq. (23).

$$\begin{aligned} F_{\mathrm{threshold}} &= \frac{2}{\pi}\left(\frac{1}{\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}X\,\mathrm{d}Y\,\mathrm{e}^{-2(X^2+Y^2)} \\ &\times \int \mathrm{d}^2\gamma\,\mathrm{e}^{-\frac{1}{2}|\gamma|^2}\mathrm{e}^{\sqrt{2}\gamma(X+\mathrm{i}Y)}\langle\gamma|m\rangle_{\mathrm{in}}\langle m|\sqrt{2}(X-\mathrm{i}Y)\rangle_{\mathrm{in}} \\ &\times \int \mathrm{d}^2\beta\,\mathrm{e}^{-\frac{1}{2}|\beta|^2}\mathrm{e}^{\sqrt{2}\beta^*(X-\mathrm{i}Y)}\langle m|\beta\rangle_{\mathrm{in}}\langle\sqrt{2}(X-\mathrm{i}Y)|m\rangle_{\mathrm{in}}, \end{aligned} \tag{24}$$

where we have used the normalized probability distribution of Eq. (19). Furthermore, the threshold fidelity is calculated by integrating out the parameters of β and γ in Eq. (24) and using the standard Aular integrating formulae,

$$F_{\mathrm{threshold}} = \frac{(2m)!}{2^{2m+1}(m!)^2}. \tag{25}$$

Eq. (25) creates the boundary between the classical and quantum domains in the teleportation of fock states. It is obviously for the fock state input the criterion of teleportation is different from that obtained for coherent state input. The threshold fidelity depends on the photon numbers of input fock states. Fig. 1 shows the dependence of the fidelity given in Eq. (25) on the photon numbers. When $m = 0$ it gives $F_{\mathrm{threshold}} = 1/2$ which is the boundary for teleportation of coher-

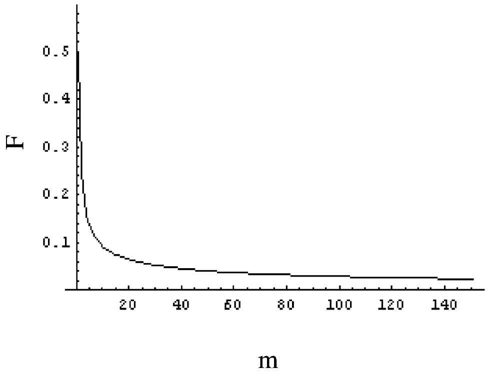

Fig. 1. The classical limit of fidelity F is plotted versus the photon numbers m of the input fock state.

ent states. In the case of $m > 0$ the fidelity always satisfies $F_{\text{threshold}} < 1/2$, especially note that for very large input photon numbers, the threshold approaches to zero, it gives the fact that the $F > 1/2$ is no longer the severe condition for quantum teleportation. For different state input there is the different criterion for evaluating the efficiency of teleportation.

4. Conclusion

In conclusion, we have analyzed the quantum teleportation of quantum states using the theory of quantum-mechanical measurement in Schrödinger picture. According to the state evolution equation, the average fidelity is deduced directly from the original definition and the criterion is created for teleportation of fock states. It is shown that the criterion is different from that for coherent state input. We believe that the results obtained in this paper are important to develop the quantum communication system since the transfer of any quantum state especially for non-classical state is inevitable in quantum communication system.

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